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POLYNOMIAL COINTEGRATION AMONG STATIONARY PROCESSES WITH LONG MEMORY *

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JEL Classification: C14, C32

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Polynomial Cointegration among Stationary Processes with Long Memory*

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In this paper we consider polynomial cointegrating relationships among stationary processes with long range dependence. We express the regression functions in terms of Hermite polynomials and we consider a form of spectral regression around frequency zero. For these estimates, we establish consistency by means of a more general result on continuously averaged estimates of the spectral density matrix at frequency zero.

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AMS classification: Primary 62M15, Secondary 62M10, 60G10

1 Introduction

The extension of the standard cointegration paradigm to more general, fractional circumstances has drawn increasing attention in the time series literature over the last decade. The possibility of fractional cointegration was already mentioned in the seminal paper by Engle and Granger (1987). Robinson (1994) was the first to establish consistency for narrow-band estimates of fractional cointegrating relationships in the stationary case. The properties of this estimator (which has become known as NBLS) were then

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investigated under nonstationary circumstances by Marinucci and Robinson (2001), Robinson and Marinucci (2001, 2003). Chen and Hurvich (2003a,b) considered principal components methods in the frequency domain, whereas Velasco (2003), Robinson and Hualde (2003) advocate pseudo-maximum likelihood methods which improve the efficiency of the estimates and yield standard asymptotic properties. Cointegration among stationary processes has also been considered, for instance by Marinucci (2000), Christensen and Nielsen (2005). Many other insightful papers on fractional cointegration have appeared in the literature, for instance Dolado and Marmol (2004), Davidson (2002).

All these papers have focused on the case of linear cointegration. Nevertheless, the possibility of polynomial cointegrating relationships seems of practical interest, for instance (but not exclusively) for applications to financial data. Nonlinear cointegration has been considered in the literature (most recently by Karlsen, Myklebust, and Tjostheim (2005)), but only in non-fractional circumstances, to the best of our knowledge. In this paper, we shall focus on nonlinear cointegrating relationships among stationary long memory processes; the restriction to a stationarity framework is made necessary by the need to exploit the rich machinery of expansions into Hermite polynomials, an extremely powerful tool to investigate nonlinear transformations (see for instance Giraitis and Surgailis (1985), Arcones (1994), Surgailis (2003)). Our general setting can be explained as follows. Let $\{A_t\} = \{x_t, e_t\}$, $t \in \mathbb{Z}$ be a stationary bivariate time series with mean zero and covariance such that

$$\mathbb{E}A_t A'_{t+\tau} := \Gamma(\tau) = \int_0^{2\pi} f(\lambda) e^{i\tau\lambda} d\lambda \ ,$$

where

$$f(\lambda) = \begin{bmatrix} f_{xx}(\lambda) & f_{xe}(\lambda) \\ f_{ex}(\lambda) & f_{ee}(\lambda) \end{bmatrix} ,$$

is the spectral density matrix of $\{A_t\}$. We shall take $\{x_t, e_t\}$ to be long memory, in the sense that

$$\gamma_{ab}(\tau) \simeq G_{ab} \tau^{d_a + d_b - 1} \tag{1}$$

for $a, b = x, e$, $0 < d_a, d_b < \frac{1}{2}$, $G_{xx}, G_{ee} > 0$, $|G_{xe}| \geq 0$. We write $z \sim I(d_z)$ for long memory processes with memory parameter d_z, and $\simeq$ to denote that the ratio of the left- and right-hand sides tends to 1.

Now assume there is a polynomial function $g(\cdot)$ such that $\mathbb{E}[g(x_t)] = 0$ and

$$y_t = g(x_t) + e_t, \qquad 0 < d_e < d_y \leq d_x < 1/2 \ ; \tag{2}$$

in this case, we say that y_t, x_t are nonlinearly cointegrated. Clearly, the standard (stationary) fractional cointegrating relationship is obtained in the special case where $g(\cdot)$ is a linear function.

Our main idea in this paper is to write $g(\cdot)$ as a sum of Hermite polynomials; the coefficients of these polynomials will be estimated by means of a spectral regression method, as Robinson (1994), Marinucci (2000) and Marinucci and Robinson (2001). We shall show that, by using a degenerating band of frequencies around the origin, then the estimator of these coefficients is consistent, even if x_t and e_t are allowed to be correlated. The plan of this paper is as follows; in Section 2 we review some results on long memory processes, Hermite polynomials and the diagram formula; in Section 3 we discuss consistent estimation of nonlinear cointegrating relationships, whereas in Section 4 we collect some comments and directions for future research. Some technical results are collected in an Appendix. In the sequel, C denotes a generic, positive, finite constant, which need not to be the same all the time it is used; for two generic matrices A and B, of equal dimension, we say that $A \simeq B$ if, for each (i,j), the ratio of the (i,j)-th elements of A and B tends to unity.

2 Nonlinear transformation of long memory process

It is well-known that, under regularity conditions, a consistent estimator of the spectral density matrix at frequency zero is given by (see for instance Hannan (1970), p.246)

$$\hat{f}_{ab}(0) = \int_{-\pi}^{\pi} K_M(\lambda) I_{ab}(\lambda) d\lambda \tag{3}$$

where the kernel $K_M(\lambda)$ is symmetric and such that $\int_{-\pi}^{\pi} K_M(\lambda)d\lambda = 1$, and M is a positive integer satisfying the bandwidth condition $1/M + M/n \to 0$. Here, $I_{ab}(\lambda)$ is the periodogram, that is

$$I_{ab}(\lambda) = \frac{1}{2\pi} \sum_{\tau=-n+1}^{n-1} c_{ab}(\tau) \exp(-i\lambda\tau) \ ,$$

for

$$c_{ab}(\tau) = \begin{cases} n^{-1} \sum_{t=1}^{n-\tau} a_t b_{t+\tau} & \tau \geq 0 \\ n^{-1} \sum_{t=|\tau|+1}^{n} a_t b_{t-|\tau|} & \tau < 0 \end{cases} \ , \ a_t, b_t \in \mathbb{R} \ , \ t = 1, 2..., n \ .$$

Equation (3) can be rewritten as

$$\hat{f}_{ab}(0) = \frac{1}{2\pi} \sum_{-n+1}^{n-1} k_M(\tau) c_{ab}(\tau) \ , \ k_M(\tau) := \int_{\pi}^{\pi} K_M(\lambda) \exp(i\tau\lambda) d\lambda \ . \tag{4}$$

As usual, we call the function $k_M(\cdot)$ a lag window, and the corresponding estimator (4) the lag window spectral density estimator. The asymptotic behaviour of $\hat{f}_{ab}(0)$ under short range dependence conditions is now standard textbook material, see for instance Taniguchi and Kakizawa (2000). Under long memory circumstances, Robinson (1994) and Lobato (1997) investigated the behaviour of, respectively, a discrete univariate and multivariate version of (3); Robinson (1994) propose an application of this statistic for the estimation of the cointegrating vector in a stationary framework. The behaviour of the lag window estimator under long memory is discussed by Marinucci (2000).

In the latter reference, the following linear cointegrating relationship is considered:

$$y_t = \beta x_t + e_t, \quad 0 \leq d_e < d_y = d_x < 1/2 \ ,$$

and it is shown that β is consistently estimated by

$$\widetilde{\beta}_M = \frac{\int_{-\pi}^{\pi} K_M(\lambda) I_{ab}(\lambda) d\lambda}{\int_{-\pi}^{\pi} K_M(\lambda) I_{bb}(\lambda) d\lambda} = \frac{\sum_{\tau=-n+1}^{n-1} k_M(\tau) c_{ab}(\tau)}{\sum_{\tau=-n+1}^{n-1} k_M(\tau) c_{bb}(\tau)} \ ,$$

under the bandwidth condition $M^2 = o(n)$ as $n \to \infty$. We consider here the same procedure but under generalized, polynomial circumstances. The investigation of nonlinear transformations requires the computation of the cumulants of the Hermite polynomials; this is usually achieved by means of the so-called diagram formula (see for instance Arcones (1994)). We introduce now the main ideas behind this approach. Consider the polynomial transformation of the Gaussian process (see also Dittmann and Granger (2002))

$$g(z_t) = \sum_{k=k_0}^{K} a_k z_t^k = \sum_{k=k_0}^{K} b_k H_k(z_t) \ , \ k_0 \geq 1 \ , \ t = 1, 2, \ldots n \ , \tag{5}$$

where

$$b_k = \frac{\mathbb{E}(g(z) H_k(z))}{k! \sigma^{2k}} \ ,$$

and $H_k(.)$ denote the well-known Hermite polynomials, which form a complete orthogonal system in the space $L^2\left(\mathbb{R}^1, (\sqrt{2\pi\sigma^2})^{-1} \exp(-z^2/2\sigma^2\right)$ of square integrable functions of Gaussian variables. These polynomials are defined through the formula:

$$H_j(z;\sigma^2) = (-1)^j \sigma^{2j} \exp\left(\frac{z^2}{2\sigma^2}\right) \frac{d^j}{dz^j} \exp\left(-\frac{z^2}{2\sigma^2}\right), \quad j = 1, 2, \ldots$$

Straightforward computation show that the first five polynomials are:

$$\begin{aligned} H_0(z) &= 1\ ,\ H_1(z) = z\ ,\ H_2(z) = z^2 - \sigma^2,\ H_3(z) = z^3 - 3z\sigma^2\ , \\ H_4(z) &= z^4 - 6z^2\sigma^2 + 3\sigma^4,\ H_5(z) = z^5 - 10z^3\sigma^2 + 15z\sigma^4\ . \end{aligned}$$

The Hermite polynomials satisfy the differential equation $kH_{k-1}(z) = dH_k(z)/dz$, under the boundary conditions $\mathbb{E}[H_m(Z)] \equiv 0$, where Z is a zero mean Gaussian variable with variance σ^2. It is also well-known that, for any mean zero Gaussian random variables v and u, we have:

$$\mathbb{E}\left[H_p(u)H_q(v)\right] = \begin{cases} p!\left[\mathbb{E}(uv)\right]^p & \text{for } p = q \\ 0 & \text{for } p \neq q \end{cases} . \tag{6}$$

The index of the first non null coefficient b_k is termed *Hermite rank* of $g(.)$. Of course, y_t is non-Gaussian unless $g(\cdot)$ is linear.

For our aims, the most important property of Hermite polynomials is their orthogonality. This property allows us to characterize in a simple way the dependence structure of a nonlinear transformation of a stationary Gaussian process that exhibit long range dependence. The following results are essentially due to Taqqu (1975, 1979) and Dobrushin and Major (1979). The non-Gaussian case is more complicated and some results using Appell polynomials are provided by Surgailis (2000). Let $z_t \sim I(d_z)$; in view of (1) and (6) it is easy to see that

$$\mathbb{E}[H_k(z_0)H_k(z_\tau)] = k!\gamma^k(\tau) \simeq C\tau^{k(2d_z-1)}, \quad \text{as } \tau \to \infty$$

so the sequence $H_{k+1}(z_t)$ is "less" dependent then $H_k(z_t)$. More precisely, if $z_t \sim I(d_z)$, then $H_k(z_t)$ can be viewed as a long memory series that is fractional integrated of order d_k ,

$$d_k := \left\{k\left(d_z - \frac{1}{2}\right) + \frac{1}{2}\right\} \vee 0 \leq d_z\ .$$

The above equation follows straightforwardly from the equality $2d_k - 1 = k(2d_z - 1)$. The fact that a nonlinear transformation of a Gaussian process with long range dependence cannot increase its memory can be used to determine the leading term of the expansion (5). For instance, assume for notational simplicity that z_t has unit variance. From (6)

$$\mathbb{E}[H_k(z_t)H_k(z_{t+\tau})] = k!\gamma^k_{zz}(\tau) = k!\int_{-\pi}^{\pi} e^{i\lambda\tau} f^{(*k)}(\lambda)d\lambda$$

where

$$f^{(*k)}(\lambda) = \int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi} f_z(\lambda-\omega_1-\dots-\omega_{k-1})f_z(\omega_1)\dots f_z(\omega_{k-1})d\omega_1\dots d\omega_{k-1}$$

is the k fold convolution of $f_z(\lambda)$ with himself (Hannan (1970), Dalla, Giratis, and Hidalgo (2004)). The convolution is defined extending $f(.)$ periodically outside of $[-\pi, \pi]$. We now have, with an obvious notation:

$$f_g(\lambda) = \sum_{k=k_0}^{K} b_k^2 k! f_z^{(*k)}(\lambda)$$

Therefore, also in the frequency domain, the rank of the transformation determines the feature of the spectral density, and for $\lambda \to 0$, the memory of $g(z_t)$.

Let us now introduce *diagrams*, which are mnemonic devices for computation of moments and cumulants of polynomial forms in Gaussian random variables. Our presentation follows Arcones (1994) and Surgailis (2003). Let p and ℓ_j , $j = 1, \ldots, p$, be given integers. A *diagram* G of order $(\ell_1, \ldots \ell_p)$ is a set of points $\{(j, \ell)) : 1 \leq j \leq p;\ 1 \leq \ell \leq \ell_j\}$ called *vertexes*, and a set of pairs of these points

$$\{((j, \ell), (k, s)) : 1 \leq j < k < p; 1 \leq \ell \leq \ell_j, 1 \leq s \leq \ell_s\} ,$$

called *edges*. Every vertex is of degree one, that is, it is considered one time for each graph. We denote by $\mathcal{V}(\ell_1, \ldots, \ell_p)$ the set of diagrams of order $(\ell_1, \cdots, \ell_p)$. The set is empty if $\ell_1 + \cdots + \ell_p$ is an odd number. The set $L_j = \{(j, \ell) : 1 \leq \ell \leq \ell_j\}$ is called the jth level of G. We will denote the set of edges of diagram G by $\mathcal{E}(G)$. Observe that edges connect vertexes of different levels (no flat edges). A diagram G is said to be *connected* if the rows of the table cannot be divided in two groups, each of which is partitioned by the diagram separately. In other words , G is connected if one cannot find a partition $P_1 \cup P_2 = \{1, \ldots, p\}$, $P_1 \cap P_2 = \emptyset$, $P_1, P_2 \neq \emptyset$, such that, for $\mathcal{E}(G) = (g_1, \ldots g_k)$, either $g_i \in \cup_{j \in P_1} L_j$ or $g_i \in \cup_{j \in P_2} L_j$ holds, for $i = 1, \ldots r$, where r is the number of edges g_i of the diagram G. The set of connected diagrams are indicated by $\mathcal{V}^c(\ell_1, \ldots, \ell_p)$. The main instrument we shall need below is the following, well known:

Diagram Formula: Let $(z_1, \ldots z_p)$ be a centered Gaussian vector, and let $\gamma_{ij} = \mathbb{E}(z_i z_j)$, $i, j = 1, \ldots, p$. Let L be a table consisting of p rows $\ell_1, \ldots \ell_p$, where ℓ_j is the order of Hermite polynomial in the variable z_j. Then

$$\begin{aligned} \mathbb{E}\left\{\prod_{j=1}^{p} H_{\ell_j}(z_j)\right\} &= \sum_{G \in \mathcal{V}(\ell_1, \ldots, \ell_p)} \prod_{1 \leq i < j \leq p} \gamma_{ij}^{\alpha_{ij}} \\ \operatorname{cum}\left(H_{\ell_1}(z_1), \ldots, H_{\ell_p}(z_p)\right) &= \sum_{G \in \mathcal{V}^c(\ell_1, \ldots, \ell_p)} \prod_{1 \leq i < j \leq p} \gamma_{ij}^{\alpha_{ij}} \end{aligned}$$

where, for each Gaussian diagram, α_{ij} is the number of edges between rows ℓ_i, ℓ_j and $\mathrm{cum}\big(H_{\ell_1}(z_1), ..., H_{\ell_p}(z_p)\big)$ represents the $p-$th order cumulant.

Examples of diagrams are represented in Figures 1 to 7 in the Appendix.

3 Nonlinear cointegration

We state here more precisely our full set of assumptions.

Assumption A

1) The following equation holds:

$$y_t = g(x_t) + e_t \ , \tag{7}$$

where for $t = 1, 2, ...$

$$\begin{aligned} g(x_t) &= \textstyle\sum_{k=k_0}^{K} a_k x_t^k = \sum_{k=k_0}^{K} b_k H_k(x_t) \ , \quad b_{k_0} \neq 0 \ , \\ e_t &= \textstyle\sum_{\widetilde{k}=\widetilde{k}_0}^{\widetilde{K}} \theta_{\widetilde{k}} \varepsilon_t^{\widetilde{k}} = \sum_{\widetilde{k}=\widetilde{k}_0}^{\widetilde{K}} \xi_{\widetilde{k}} H_{\widetilde{k}}(\varepsilon_t) \ , \quad \xi_{\widetilde{k}_0} \neq 0 \ , \end{aligned}$$

2) $(x_t, \varepsilon_t)'$ are jointly Gaussian and long memory, that is, as $\tau \to \infty$

$$\begin{aligned} \gamma_{xx}(\tau) &\simeq G_{xx}\tau^{2d_x-1}, \ 0 < G_{xx} < \infty \\ \gamma_{\varepsilon\varepsilon}(\tau) &\simeq G_{\varepsilon\varepsilon}\tau^{2d_\varepsilon-1}, \ 0 < G_{\varepsilon\varepsilon} < \infty \\ \gamma_{x\varepsilon}(\tau) &\simeq G_{x\varepsilon}\tau^{d_x+d_\varepsilon-1}, \ |G_{x\varepsilon}| < \infty \end{aligned}$$

for $0 \leq d_\varepsilon, d_x < \frac{1}{2}$.

3) The parameters $K, \widetilde{k}_0$ are such that

$$K(2d_x - 1) > \left\{-1 \vee \widetilde{k}_0(2d_\varepsilon - 1)\right\} \ .$$

Assumptions A1-A2 identify a polynomial cointegration model where the residual is a Gaussian subordinated process. Assumption A3 ensures that $H_K(x_t)$ is still a long memory process, with stronger memory than e_t. This is needed for consistency and indeed it is also a necessary identification condition: recall x_t and e_t can be correlated, so there are no means to distinguish $H_k(x_t)$ and e_t unless the former has stronger long range dependence. Recall that $H_k(x_t) \sim I(d_k)$, $k = k_0, ..., K$, where $2d_k - 1 := k(2d_x - 1)$. In this paper, we take k_0 and K to be known, whereas their estimation will be addressed in a different work. Note that to implement our estimates we need no a priori information on $\widetilde{k}_0, \widetilde{K}$, although the value of $\widetilde{k}_0(2d_\varepsilon - 1)$ does affect the rate of consistency of our estimators.

As mentioned before, (2) is a cointegrating relation, so we allow $\mathbb{E}(x_t \varepsilon_t)$ (and hence $\mathbb{E}(x_t e_t)$) to be different from zero. As for linear cointegration, this leads to the inconsistency of OLS and justifies the use of the spectral regression methods for the estimation of Hermite coefficients. Concerning the kernel, we write $k_M(\cdot) = k(\tau/M)$ and introduce the following

Assumption B: The kernel $k(\cdot)$ is a real-valued, symmetric Lebesgue measurable function that, for $v \in \mathbb{R}$, satisfies

$$\int_{-1}^{1} k(v)dv = 1 \qquad 0 \leq k(v) \leq \infty, \quad k(v) = 0 \quad \text{for } |v| > 1.$$

Our final assumption is a standard bandwidth condition.

Assumption C: Let $\eta = K \vee \widetilde{k}_0$; as $n \to \infty$,

$$\frac{1}{M} + \frac{M^{3\vee(\eta-2)}}{n} \to 0 \ .$$

Assumption C imposes a minimal lower bound and a significant upper bound on the behaviour of the user-chosen bandwidth parameter M. The need for this bandwidth condition is made clear by inspection of the proof in the appendix; heuristically, as K grows the signal in $H_K(x_t)$ decreases, which makes the estimation harder; on the other hand an increase in $\widetilde{k}_0$ makes the convergence rates in Lemma 1 and Theorem 1 faster, whence the need for tighter bandwidth conditions. We are not claiming Assumption C is sharp, however an inspection of the Proof of Lemma 1 reveals that any improvement is likely to require at least almost unmanageable computations.

Equation (7) can be rewritten more compactly as

$$y_t = \boldsymbol{\beta}' H(x_t) + e_t \text{ , where } H(x_t) = \{H_1(x_t), ..., H_K(x_t)\}' \ .$$

Let us now define:

$$f_{HH}(\lambda) = \begin{bmatrix} f_{11}(\lambda) & 0 & \cdots & \cdots \\ 0 & f_{22}(\lambda) & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & f_{KK}(\lambda) \end{bmatrix} , \ f_{He}(\lambda) = \begin{bmatrix} f_{1e}(\lambda) \\ f_{2e}(\lambda) \\ \vdots \\ f_{Ke}(\lambda) \end{bmatrix}$$

and let also, for $a, b = 1, 2, \ldots K$.

$$\begin{aligned}
\gamma_{ab}(\tau) &= \mathbb{E}\left[H_a(x_t)H_b(x_{t+\tau})\right] = a!\delta_a^b \left\{\mathbb{E}\left(x_t x_{t+\tau}\right)\right\}^a , \\
\gamma_{ae}(\tau) &= \mathbb{E}\left[H_a(x_t)e_{t+\tau}\right] = \mathbb{E}\left[H_a(x_t) \sum_{\widetilde{k}=\widetilde{k}_0}^{\widetilde{K}} \xi_{\widetilde{k}} H_{\widetilde{k}}(\varepsilon_t)\right] \\
&= \left\{ \begin{array}{c} a!\xi_a \left\{\mathbb{E}\left(x_t \varepsilon_{t+\tau}\right)\right\}^a \text{ for } a \leq \widetilde{K} \\ 0 \text{ , otherwise} \end{array} \right. .
\end{aligned}$$

where δ_a^b represents the Kronecker delta function. Likewise

$$\begin{aligned}
f_{aa}(\lambda) &: = (2\pi)^{-1} \int_{-\infty}^{\infty} \gamma_{aa}(\tau) e^{-i\lambda\tau} = a! f_x^{(*a)}(\lambda) , \\
f_{ay}(\lambda) &: = (2\pi)^{-1} \int_{-\infty}^{\infty} \gamma_{ay}(\tau) e^{-i\lambda\tau} , \ \gamma_{ay}(\tau) := \mathbb{E}\left[H_a(x_t) y_{t+\tau}\right] \\
f_{ae}(\lambda) &: = (2\pi)^{-1} \int_{-\infty}^{\infty} \gamma_{ae}(\tau) e^{-i\lambda\tau} .
\end{aligned}$$

The Weighted Covariance Estimator (WCE) of $\beta' = (\beta_1, \ldots, \beta_K)$ is defined as

$$\hat{\beta}_M = \hat{f}_{HH}(0)^{-1} \hat{f}_{Hy}(0) ,$$

whence

$$\hat{\beta}_M - \beta = \hat{f}_{HH}(0)^{-1} \hat{f}_{He}(0) ;$$

as usual, we assume $\hat{f}_{HH}(0)$ is non-singular, where

$$\hat{f}_{HH}(0) = \frac{1}{2\pi} \left[\begin{array}{ccc} \sum_{\tau=-M}^{M} k(\tau/M) c_{11}(\tau) & \cdots & \sum_{\tau=-M}^{M} k(\tau/M) c_{1K}(\tau) \\ \vdots & \ddots & \vdots \\ \sum_{\tau=-M}^{M} k(\tau/M) c_{K1}(\tau) & \cdots & \sum_{\tau=-M}^{M} k(\tau/M) c_{KK}(\tau) \end{array} \right] ,$$

$$\hat{f}_{Hz}(0) = \frac{1}{2\pi} \left[\begin{array}{c} \sum_{\tau=-M}^{M} k(\tau/M) c_{1z}(\tau) \\ \vdots \\ \sum_{\tau=-M}^{M} k(M) c_{Kz}(\tau) \end{array} \right]$$

and

$$\begin{aligned}
c_{ab}(\tau) &= \left\{ \begin{array}{ll} n^{-1} \sum_{t=1}^{n-\tau} H_a(x_t) H_b(x_{t+\tau}) & \tau \geq 0 \\ n^{-1} \sum_{t=|\tau|+1}^{n} H_a(x_t) H_b(x_{t-|\tau|}) & \tau < 0 \end{array} \right. \\
c_{az}(\tau) &= \left\{ \begin{array}{ll} n^{-1} \sum_{t=1}^{n-\tau} H_a(x_t) z_{t+\tau} & \tau \geq 0 \\ n^{-1} \sum_{t=|\tau|+1}^{n} H_a(x_t) z_{t-|\tau|} & \tau < 0 \end{array} \right.
\end{aligned}$$

for $a, b = 1, 2, \ldots K$, $z = e, y$. The following lemma is the main tool for our consistency result, compare Lemma 1 in Marinucci (2000). As before, we write

$$d_a := a\left(d_x - \frac{1}{2}\right) + \frac{1}{2}\ ,\ d_e = \left\{\widetilde{k}_0\left(d_\varepsilon - \frac{1}{2}\right) + \frac{1}{2}\right\} \vee 0\ ;$$

by Assumption A3 we have $d_a > 0$, $a = k_0, ..., K$.

LEMMA 1 Under Assumptions A-C, as $n \to \infty$ we have:

$$\sum_{\tau=-M}^{M} k\left(\frac{\tau}{M}\right)\{c_{ab}(\tau) - \gamma_{ab}(\tau)\} = o_p(M^{d_a+d_b}) \tag{8}$$

$$\sum_{\tau=-M}^{M} k\left(\frac{\tau}{M}\right)\{c_{ae}(\tau) - \gamma_{ae}(\tau)\} = o_p(M^{d_a+d_e}) \tag{9}$$

for $a, b = 1, 2, \ldots K$

Proof See Appendix

We are now ready to state the main result of this paper. Let

$$B_{ab} \ :\ = a! G_{xx}^a \delta_a^b \int_{-1}^{1} k(\upsilon)|\upsilon|^{a(2d_x-1)} d\upsilon < \infty\ ,$$

$$B_{ae} \ :\ = a! \xi_a \{G_{x\varepsilon}\}^a \int_{-1}^{1} k(\upsilon)|\upsilon|^{a(d_x+d_\varepsilon-1)} d\upsilon < \infty\ , \text{ for } a \leq \widetilde{K}\ ,$$

see also Assumption B, $a, b = k_0, ..., K$. Let

$$\mathcal{B}_{HH} = \operatorname{diag}\{B_{11}, \ldots B_{KK}\}\ ,\ \mathcal{B}_{He} = \{B_{1e}, ..., B_{Ke}\}\ ,\ \mathcal{M} = \operatorname{diag}\left\{M^{-d_1}, \ldots M^{-d_K}\right\}\ .$$

Note that $B_{ae} = 0$ unless $a \leq \widetilde{K}$, due to the orthogonality of Hermite polynomials.

Theorem 1 Under the Assumptions A-C, as $n \to \infty$

$$\begin{bmatrix} M^{d_1-d_e} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & M^{d_K-d_e} \end{bmatrix}\left(\hat{\beta}_M - \beta\right) = \mathcal{B}_{HH}^{-1}\mathcal{B}_{He} + o_p(1)\ .$$

Proof By the dominated convergence theorem, as $M \to \infty$

$$M^{-(d_a+d_b)}\sum_{\tau=-M}^{M} k\left(\frac{\tau}{M}\right)\gamma_{ab}(\tau) = \sum_{\tau=-M}^{M} k\left(\frac{\tau}{M}\right)\frac{\gamma_{ab}(\tau)}{M^{d_a+d_b-1}}\frac{1}{M} \to B_{ab}$$

$$M^{-(d_a+d_e)}\sum_{\tau=-M}^{M} k\left(\frac{\tau}{M}\right)\gamma_{ae}(\tau) = \sum_{\tau=-M}^{M} k\left(\frac{\tau}{M}\right)\frac{\gamma_{1e}(\tau)}{M^{d_a+d_e-1}}\frac{1}{M} \to B_{ae}$$

From Lemma 1, it follows easily that

$$\hat{f}_{HH}(0) = \begin{bmatrix} \zeta_1 + o_p(M^{2d_1}) & o_p(M^{d_1+d_2}) & \cdots & o_p(M^{d_1+d_p}) \\ o_p(M^{d_2+d_1}) & \zeta_2 + o_p(M^{2d_2}) & \cdots & o_p(M^{d_2+d_p}) \\ \vdots & \vdots & \ddots & \vdots \\ o_p(M^{d_K+d_1}) & \cdots & \cdots & \zeta_K + o_p(M^{2d_K}) \end{bmatrix}$$

where

$$\zeta_a := \frac{1}{2\pi} \sum_{\tau=-M}^{M} k(\frac{\tau}{M})\gamma_{aa}(\tau) \ .$$

Moreover

$$\mathcal{M}\hat{f}_{HH}(0)\mathcal{M} = \begin{bmatrix} B_{11} + o_p(1) & \cdots & o_p(1) \\ \vdots & \ddots & \vdots \\ o_p(1) & \cdots & B_{KK} + o_p(1) \end{bmatrix} \to \mathcal{B}_{HH} \ .$$

Therefore, for $M \to \infty$

$$\hat{f}_{HH}(0) = \mathcal{M}^{-1}\mathcal{B}_{HH}\mathcal{M}^{-1} + o_p(1) = \mathcal{B}_{HH}\mathcal{M}^{-2} + o_p(1) \ ,$$

since $\mathcal{B}_{HH}$ is diagonal and hence commutes with $\mathcal{M}^{-1}$. Using the same arguments, it follows easily that:

$$M^{-d_e}\mathcal{M}\hat{f}_{he}(0) \to \mathcal{B}_{He} \ , \text{ as } n \to \infty \ .$$

Finally, as $n \to \infty$,

$$M^{-d_e}\mathcal{M}^{-1}\left\{\hat{\beta}_M - \beta\right\} = \left\{\mathcal{M}\hat{f}_{hh}(0)\mathcal{M}\right\}^{-1} \mathcal{M}M^{-d_e}\hat{f}_{he}(0) \to \mathcal{B}_{HH}^{-1}\mathcal{B}_{He} \ ,$$

which completes the proof of Theorem 1.

□

Remark In Theorem 1 we have proved the consistency of the WCE estimator of the cointegrating vector, $\hat{\beta}_M \xrightarrow{p} \beta$. In a very loose sense, this result follows from consistency of a continuously averaged estimate of the spectral density at frequency zero, see Lemma 1. It is also possible to use Lemma 1 to derive a robust estimate for the memory parameter of an observed, Gaussian subordinated series $w_t := g(x_t)$, $(k_0(d_x - \frac{1}{2}) + \frac{1}{2} =: d_w$, say). We use a very similar idea to the averaged periodogram estimate advocated by Robinson (1994). More precisely, with an obvious notation we can consider

$$\begin{aligned} \widetilde{d}_w \quad : \quad &= \frac{\log\left|\sum_{\tau=-M}^{M} k(\frac{\tau}{M})c_{ww}(\tau)\right|}{2\log M} = d_w + \frac{\log B_{ww}}{2\log M} + o_p(1) \ , \\ &= d_w + o_p(1) \ , \end{aligned}$$

where we have used Lemma 1. This estimate converges at a mere logarithmic rate and it is not asymptotically centered around zero; it is however consistent under much broader circumstances than usually allowed for in the literature. See also Dalla, Giratis, and Hidalgo (2004) for very general results on consistency for long memory estimates.

4 Comments and conclusions

We view this paper as a first step in a new research direction, and as such we are well aware that it leaves several questions unresolved and open for future research. A first issue relates to the choice of the Hermite rank k_0 and of K. As far as the former is concerned, we remark that for the great majority of practical applications, k_0 can be taken a priori as 1 or 2. Under the assumption that $k_0 = 1$, the equality $d_x = d_y$ holds; this trivial observation immediately suggests a naive test for $k_0 = 1$, which can be simply implemented by testing for equality of the two memory parameters. It should be noted, however, that when x_t and y_t are cointegrated the standard asymptotic results on multivariate long memory estimation (for instance Robinson (1995)) do not hold. Incidentally, we note that the nonlinear framework allows to cover the possibility of cointegration among time series with different integration orders, a significant extension over the standard paradigm.

For K, we can take as an identifying assumption

$$K := \operatorname{argmax}(k : k(2d_x - 1) > (2d_e - 1)) \; ; \tag{10}$$

higher order terms can be thought of as included by definition in the residuals, to make identification possible. Indeed, it is natural to suggest to view $g(.)$ as a general nonlinear function and envisage K as growing with n; we expect, however, that only the projection coefficients b_k with k satisfying (10) could be consistently estimated in this broader framework. On the other hand, we note that the it is also possible to estimate consistently $K^* < K$ regression coefficients, by simply dropping the higher order regressors: it is immediate to see that their inclusion in the residual would not alter any of our asymptotic result (there may be an effect in finite samples, however). We stress that a lower number of regressors allows in general a weaker bandwidth condition, see Assumption C.

The extension to multivariate regressors does not seem to pose any new theoretical problem: multivariate generalizations of Hermite expansions are well known to the literature. Of course, much more challenging seems to be the possibility to allow for multiple cointegrating relationships. An important point to remark is the following. In standard cointegration theory, the role of the variables on the left and on on the right-hand sides is, by all means, symmetric: this is no longer the case when nonlinear relationships are allowed. In particular, it should be noted that the memory parameter of the dependent variable y_t is always smaller or equal than d_x; this information can be exploited in an obvious way to decide the form of the regression, provided that first step estimates of the long memory parameters are available. We also remark that our procedure requires a preliminary knowledge on the variance of the regressor x_t; such knowledge can clearly be derived from first step estimates, and we leave for future research the analysis of its consequences in finite samples.

In this paper, we restricted ourselves to consistency results, and gave no hint on asymptotic distributions. The latter are likely to be non-Gaussian, at least if the Hermite rank is larger than one and/or the memory of the raw series is such to make their autocovariances not square summable (see for instance Fox and Taqqu (1985, 1986). A much wider issue relates to the possible extension to nonstationary circumstances. Here, a major technical difficulty arises: the higher order terms in Hermite expansions need no longer be of smaller order in the presence of nonstationarity. We believe, however, that the stationary framework considered in this paper is of sufficient interest by itself for applications to real data, see again Christensen and Nielsen (2005) for examples on how fractional cointegration among stationary variables may be implied by some models of volatility, based on the Black-Scholes formula for option pricing.

Appendix

Proof of Lemma 1 Recall we have

$$\gamma_{ab}(\tau) \simeq G|\tau|^{d_a+d_b-1} \quad \text{as} \quad \tau \to \infty \ ,$$

where for $a, b = 1, ..., K$, d_a is such that

$$d_a := \begin{cases} \frac{a}{2}(2d_x - 1) + \frac{1}{2} & \text{for} \quad a(2d_x - 1) > -1 \\ 0 & \text{for} \quad a(2d_x - 1) < -1 \end{cases} . \tag{11}$$

The first part of the proof follows closely Marinucci (2000). For (8), it is sufficient to show that

$$\begin{aligned} Var\left\{ \sum_{\tau=-M+1}^{M-1} k\left(\frac{\tau}{M}\right) c_{ab}(\tau) \right\} &= \mathbb{E}\left\{ \sum_{\tau=-M+1}^{M-1} k\left(\frac{\tau}{M}\right) \left[c_{ab}(\tau) - \left(1 - \frac{\tau}{n}\right) \gamma_{ab}(\tau) \right] \right\}^2 \\ &\leq C \sum_{p=-M}^{M} \sum_{q=-M}^{M} |Cov\{c_{ab}(p), c_{ab}(q)\}| = o(M^{2d_a+2d_b}) \end{aligned}$$

From Hannan (1970), p.210 we have:

$$
\begin{aligned}
& Cov\{c_{ab}(p), c_{ab}(q)\} \\
= \; & \frac{1}{n} \sum_{r=-n+1}^{n-1} \left(1 - \frac{|r|}{n}\right) \{\gamma_{aa}(r)\gamma_{bb}(r+q-p) + \gamma_{ab}(r+q)\gamma_{ba}(r-p)\} \quad (12) \\
& + \frac{1}{n^2} \sum_{r=-n+1}^{n-1} \sum_{s=1-r}^{n-r} \mathrm{cum}_{abab}\,(s, s+p, s+r, s+r+q) \ , \quad (13)
\end{aligned}
$$

where

$$\mathrm{cum}_{abab}\,(s, s+p, s+r, s+r+q) = \mathrm{cum}\,\{H_a(x_s), H_b(x_{s+p}), H_a(x_{s+r}), H_b(x_{s+r+q})\} \ .$$

Likewise, for (9) we shall show that

$$
\begin{aligned}
Var\left\{\sum_{\tau=-M+1}^{M-1} k\left(\frac{\tau}{M}\right) c_{ae}(p)\right\} &= \mathbb{E}\left\{\sum_{\tau=-M+1}^{M-1} k\left(\frac{\tau}{M}\right)\left[c_{ae}(\tau) - \left(1 - \frac{\tau}{n}\right)\gamma_{ae}(\tau)\right]\right\}^2 \\
&\leq C \sum_{p=-M}^{M} \sum_{q=-M}^{M} |Cov\{c_{ae}(p), c_{ae}(q)\}| = o(M^{2d_a+2d_e})
\end{aligned}
$$

where

$$
\begin{aligned}
& Cov\,\{c_{ae}(p), c_{ae}(q)\} \\
= \; & \frac{1}{n} \sum_{r=-n+1}^{n-1} \left(1 - \frac{|r|}{n}\right) \{\gamma_{aa}(r)\gamma_{ee}(r+q-p) + \gamma_{ae}(r+q)\gamma_{ea}(r-p)\} \quad (14) \\
& + \frac{1}{n^2} \sum_{r=-n+1}^{n-1} \sum_{s=1-r}^{n-r} \mathrm{cum}_{aeae}\,(s, s+p, s+r, s+r+q) \ , \quad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathrm{cum}_{aeae}\,(s, s+p, s+r, s+r+q) \\
= \; & \mathrm{cum}\,\{H_a(x_s), e_{s+p}, H_a(x_{s+r}), e_{s+r+q}\} \\
= \; & \sum_{k=\widetilde{k}_0}^{\widetilde{K}} \sum_{k'=\widetilde{k}_0}^{\widetilde{K}} \mathrm{cum}\,\{H_a(x_s), H_k(\varepsilon_{s+p}) H_a(x_{s+r}), H_{k'}, (\varepsilon_{s+r+q})\} \ .
\end{aligned}
$$

The argument for (12) and (14) is the same. For instance, for (12) we have

$$
\begin{aligned}
& \sum_{p=-M}^{M} \sum_{q=-M}^{M} \frac{1}{n} \left| \sum_{r=-n+1}^{n-1} \left(1 - \frac{|r|}{n}\right) \{\gamma_{aa}(r)\gamma_{bb}(r+q-p)\} \right| \\
\leq \; & C\frac{M}{n} \sum_{\tau=-2M}^{2M} \left(\sum_{|r|\leq 2M} (|r|+1)^{2d_a-1}(|r+\tau|+1)^{2d_b-1} + \right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{|r|>2M} (|r|+1)^{2d_a-1}(|r+\tau|+1)^{2d_b-1}\Bigg) \\
&= C\frac{M}{n}\left[\sum_{|r|\le 2M}\left((|r|+1)^{2d_a-1}\sum_{\tau=-2M}^{2M}(|r+\tau|+1)^{2d_b-1}\right)\right. \\
&\quad \left. + \sum_{\tau=-2M}^{2M}\left(\sum_{2M<|r|<n}(|r|+1)^{2d_a-1}(|r+\tau|+1)^{2d_b-1}\right)\right] \\
&= O(Mn^{-1}M^{2d_a}M^{2d_b}) + O(M^2n^{-1}n^{2d_a+2d_b-1}) = o(M^{2d_a+2d_b}) \ .
\end{aligned}
$$

As usual, summations over empty sets are taken to be equal to zero. For the second term we have:

$$
\begin{aligned}
&\sum_{p=-M}^{M}\sum_{q=-M}^{M}\frac{1}{n}\left|\sum_{r=-n+1}^{n-1}\left(1-\frac{|r|}{n}\right)\gamma_{ab}(r+p)\gamma_{ba}(r-q)\right| \\
\le\ & C\sum_{p=-M}^{M}\sum_{q=-M}^{M}\frac{1}{n}\sum_{r=-n+1}^{n-1}\left(1-\frac{|r|}{n}\right)\frac{1}{2}\left|\gamma_{ab}^2(r+p)+\gamma_{ba}^2(r-q)\right| \\
\le\ & \frac{C}{n}\sum_{|r|\le 2M}\left[\sum_{p=-M}^{M}(|r+p|+1)^{2d_a+2d_b-2}+\sum_{q=-M}^{M}(|r-q|+1)^{2d_a+2d_b-2}\right] \\
& +C\frac{M^2}{n}\sum_{2M<|r|<n}\left[(|r+p|+1)^{2d_a+2d_b-2}+(|r-q|+1)^{2d_a+2d_b-2}\right] \\
=\ & O(Mn^{-1}M^{2d_a+2d_b}) + O(M^2n^{-1}n^{2d_a+2d_b-1}) = o(M^{2d_a+2d_b}) \ .
\end{aligned}
$$

The orders of magnitude of the cumulants are investigated by means of the diagram formula. The proof is quite tedious. We shall focus on (15) as the argument for (13) is entirely analogous. From the diagram formula it follows easily that, for any finite $k, k' \ge \widetilde{k}_0$

$$
\begin{aligned}
&\sum_{k=\widetilde{k}_0}^{\widetilde{K}}\sum_{k'=\widetilde{k}_0}^{\widetilde{K}}\left|\mathrm{cum}\left\{H_a(x_s), H_k(\varepsilon_{s+p}), H_a(x_{s+r}), H_{k'}(\varepsilon_{s+r+q})\right\}\right| \\
\le\ & C\left|\mathrm{cum}\left\{H_a(x_s), H_{\widetilde{k}_0}(\varepsilon_{s+p}), H_a(x_{s+r}), H_{\widetilde{k}_0}(\varepsilon_{s+r+q})\right\}\right| . \qquad (16)
\end{aligned}
$$

Indeed, increasing the value of $\widetilde{k}_0$ to k, k' entails including more products of covariances in the cumulant, and these covariances are bounded. In order to simplify the presentation, we divide it in three parts, that is

1) $a = 1$, $\widetilde{k}_0 \ge 2$ or $a \ge 2$, $\widetilde{k}_0 = 1$
2) $a = 2$, $\widetilde{k}_0 \ge 2$ or $a \ge 2$, $\widetilde{k}_0 = 2$

3) $a, \widetilde{k}_0 \geq 3$.

Throughout the proof, we shall assume for brevity's sake $\widetilde{k}_0(2d_\varepsilon - 1) > -1$; it is simple to check that for $\widetilde{k}_0(2d_\varepsilon - 1) \leq -1$ the proof is analogous, indeed slightly simpler.

Part I: $a = 1$, $\widetilde{k}_0 \geq 2$ **or** $a \geq 2$, $\widetilde{k}_0 = 1$

For $a = 1$, $\widetilde{k}_0 = 2$ we have

$$\begin{aligned}
& \sum_{p=-M}^{M} \sum_{q=-M}^{M} \frac{1}{n^2} \left| \operatorname{cum} \{x_s, H_2(\varepsilon_{s+p}), x_{s+r}, H_2(\varepsilon_{s+r+q})\} \right| \\
\leq & \sum_{p=-M}^{M} \sum_{q=-M}^{M} \frac{C}{n^2} \Bigg| \sum_{r=-n+1}^{n-1} \sum_{s=1-r}^{n-r} \gamma_{x\varepsilon}(p)\gamma_{x\varepsilon}(q)\gamma_{\varepsilon\varepsilon}(r+q-p) \\
& + \gamma_{\varepsilon\varepsilon}(r+q-p)\gamma_{\varepsilon x}(r-p)\gamma_{x\varepsilon}(r+q) \Bigg| \\
\leq & \frac{C}{n} \sum_{p=-M}^{M} (|p|+1)^{d_x+d_\varepsilon-1} \sum_{q=-M}^{M} (|q|+1)^{d_x+d_\varepsilon-1} \left(\sum_{|r|\leq 3M} (|r+q-p|+1)^{2d_\varepsilon-1} \right. \\
& \left. + \sum_{3M<|r|\leq n} (|r+q-p|+1)^{2d_\varepsilon-1} \right) \\
& + \frac{C}{n} \sum_{p=-M}^{M} \sum_{q=-M}^{M} \left(\sum_{|r|\leq 3M} (|r+q-p|+1)^{2d_\varepsilon-1} (|r-p|+1)^{d_x+d_\varepsilon-1} (|r+q|+1)^{d_x+d_\varepsilon-1} \right. \\
& \left. + \sum_{3M<|r|\leq n} (|r+q-p|+1)^{2d_\varepsilon-1} (|r-p|+1)^{d_x+d_\varepsilon-1} (|r+q|+1)^{d_x+d_\varepsilon-1} \right) \\
= & O(n^{-1} M^{d_x+d_\varepsilon} M^{d_x+d_\varepsilon} M^{2d_\varepsilon}) + O(n^{-1} M^{2d_x+2d_\varepsilon} n^{2d_\varepsilon}) \\
& + O(n^{-1} M M^{d_x+d_\varepsilon} M^{2d_\varepsilon}) + O(n^{-1} M^2 n^{2d_x+4d_\varepsilon-3}) \\
= & O\left(\frac{M}{n} M^{2d_x+4d_\varepsilon-1} \right) + o(M^{4d_\varepsilon+2d_x}) + O\left(\frac{M^2}{n} M^{d_x+3d_\varepsilon-1} \right) + o(M^{4d_\varepsilon+2d_x} \Big) \\
= & o(M^{2d_x+4d_\varepsilon-1}) = o(M^{2d_x+2d_e}) \quad \text{ because } 2d_e = 4d_\varepsilon - 1 \ .
\end{aligned}$$

The extension to $\widetilde{k}_0 > 2$ is trivial:

$$\begin{aligned}
& \operatorname{cum} \left\{ x_s, H_{\widetilde{k}_0}(\varepsilon_{s+p}), x_{s+r}, H_{\widetilde{k}_0}(\varepsilon_{s+r+q}) \right\} \\
= & \sum_{p=-M}^{M} \sum_{q=-M}^{M} \frac{C}{n^2} \Bigg| \sum_{r=-n+1}^{n-1} \sum_{s=1-r}^{n-r} \gamma_{x\varepsilon}(p)\gamma_{x\varepsilon}(q)\gamma_{\varepsilon\varepsilon}^{\widetilde{k}_0-1}(r+q-p)
\end{aligned}$$

$$
\begin{aligned}
& +\gamma_{\varepsilon\varepsilon}^{\widetilde{k}_0-1}(r+q-p)\gamma_{\varepsilon x}(r-p)\gamma_{x\varepsilon}(r+q)\Big| \\
= \ & O(n^{-1}M^{2d_x+2d_\varepsilon}M^{(\widetilde{k}_0-1)(2d_\varepsilon-1)+1})+O(n^{-1}M^{2d_x+2d_\varepsilon}n^{(\widetilde{k}_0-1)(2d_\varepsilon-1)+1}) \\
& +O(n^{-1}M^{d_x+d_\varepsilon+1}M^{(\widetilde{k}_0-1)(2d_\varepsilon-1)+1})+O(n^{-1}M^{2}n^{2d_x+2d_\varepsilon-2+(\widetilde{k}_0-1)(2d_\varepsilon-1)+1}) \\
= \ & O\left(\frac{M}{n}M^{2d_x+2d_\varepsilon-1}M^{(\widetilde{k}_0-1)(2d_\varepsilon-1)+1}\right)+o(M^{2d_x+\widetilde{k}_0(2d_\varepsilon-1)+1}) \\
& +O\left(\frac{M^2}{n}M^{d_x+d_\varepsilon-1}M^{(\widetilde{k}_0-1)(2d_\varepsilon-1)+1}\right)+O(n^{-1}M^{2}n^{2d_x+\widetilde{k}_0(2d_\varepsilon-1)}) \\
= \ & o(M^{2d_x+2d_e}),
\end{aligned}
$$

by the same argument as before. The proof for $a \geq 2$, $\widetilde{k}_0 = 1$ is entirely analogous and hence omitted.

Part II: $a = 2$, $\widetilde{k}_0 \geq 2$ **or** $a \geq 2$, $\widetilde{k}_0 = 2$

For $a = 2$, $\widetilde{k}_0 = 2$ we have

$$
\begin{aligned}
& \sum_{p=-M}^{M}\sum_{q=-M}^{M}\frac{1}{n^2}\left|\sum_{r=-n+1}^{n-1}\sum_{s=1-r}^{n-r}\operatorname{cum}\{H_2(x_s)H_2(\varepsilon_{s+p})H_2(x_{s+r})H_2(\varepsilon_{s+r+q})\}\right| \\
\leq \ & \sum_{p=-M}^{M}\sum_{q=-M}^{M}\frac{C}{n^2}\Big|\sum_{r=-n+1}^{n-1}\sum_{s=1-r}^{n-r}\gamma_{x\varepsilon}(p)\gamma_{x\varepsilon}(q)\gamma_{xx}(r)\gamma_{\varepsilon\varepsilon}(r+q-p) \\
& +\gamma_{x\varepsilon}(p)\gamma_{x\varepsilon}(r+q)\gamma_{\varepsilon x}(r-p)\gamma_{x\varepsilon}(q) \\
& +\gamma_{xx}(r)\gamma_{\varepsilon\varepsilon}(r+q-p)\gamma_{\varepsilon x}(r-p)\gamma_{x\varepsilon}(r+q)\Big| \\
\leq \ & \frac{C}{n}\Bigg\{\sum_{p=-M}^{M}\sum_{q=-M}^{M}\sum_{|r|\leq 3M}\Big[(|p|+1)^{d_x+d_\varepsilon-1}(|q|+1)^{d_x+d_\varepsilon-1}(|r|+1)^{2d_x-1}(|r+q-p|+1)^{2d_\varepsilon-1} \\
& +(|p|+1)^{d_x+d_\varepsilon-1}(|q|+1)^{d_x+d_\varepsilon-1}(|r+q|+1)^{d_x+d_\varepsilon-1}(|r-p|+1)^{d_x+d_\varepsilon-1} \\
& + (|r|+1)^{2d_x-1}(|r+q-p|+1)^{2d_\varepsilon-1}(|r-p|+1)^{d_x+d_\varepsilon-1}(|r+q|+1)^{d_x+d_\varepsilon-1}\Big]\Bigg\} \\
& +\frac{C}{n}\Bigg\{\sum_{p=-M}^{M}\sum_{q=-M}^{M}\sum_{3M<r\leq n}\Big[(|p|+1)^{d_x+d_\varepsilon-1}(|q|+1)^{d_x+d_\varepsilon-1}(|r|+1)^{2d_x-1}(|r+q-p|+1)^{2d_\varepsilon-1} \\
& +(|p|+1)^{d_x+d_\varepsilon-1}(|q|+1)^{d_x+d_\varepsilon-1}(|r+q|+1)^{d_x+d_\varepsilon-1}(|r-p|+1)^{d_x+d_\varepsilon-1} \\
& + (|r|+1)^{2d_x-1}(|r+q-p|+1)^{2d_\varepsilon-1}(|r-p|+1)^{d_x+d_\varepsilon-1}(|r+q|+1)^{d_x+d_\varepsilon-1}\Big]\Bigg\} \\
= \ & O(n^{-1}M^{2d_x+2d_\varepsilon}M^{2d_\varepsilon})+O(n^{-1}M^{d_x+d_\varepsilon}M^{d_x+d_\varepsilon}M^{d_x+d_\varepsilon})+O(n^{-1}M^{3d_\varepsilon+3d_x}) \\
& +O(n^{-1}M^{2d_x+2d_\varepsilon}n^{2d_x+2d_\varepsilon-1})+O(n^{-1}M^{2d_x+2d_\varepsilon}n^{2d_x+2d_\varepsilon-1})+O(M^2n^{-1}n^{4d_x+4d\varepsilon-3})
\end{aligned}
$$

$$
\begin{aligned}
&= O\left(\frac{M^2}{n}M^{2d_x+4d_\varepsilon-2}\right)+O\left(\frac{M^2}{n}M^{2d_x+4d_\varepsilon-2}\right)+o(M^{4d_x+4d_\varepsilon-2})\\
&= o(M^{2d_2+2d_e}) \text{ because } 2d_2=4d_x-1 \text{ and } 2d_e=4d_\varepsilon-1\ .
\end{aligned}
$$

For $\widetilde{k}_0>2$ the argument is very much the same:

$$
\begin{aligned}
&\sum_{p=-M}^{M}\sum_{q=-M}^{M}\frac{1}{n^2}\left|\sum_{r=-n+1}^{n-1}\sum_{s=1-r}^{n-r}\operatorname{cum}\{H_2(x_s)H_{\widetilde{k}_0}(\varepsilon_{s+p})H_2(x_{s+r})H_{\widetilde{k}_0}(\varepsilon_{s+r+q})\}\right|\\
\le\ &\sum_{p=-M}^{M}\sum_{q=-M}^{M}\frac{C}{n^2}\Bigg|\sum_{r=-n+1}^{n-1}\sum_{s=1-r}^{n-r}\gamma_{x\varepsilon}(p)\gamma_{x\varepsilon}(q)\gamma_{xx}(r)\gamma_{\varepsilon\varepsilon}^{\widetilde{k}_0-1}(r+q-p)\\
&+\gamma_{x\varepsilon}(p)\gamma_{x\varepsilon}(q)\gamma_{x\varepsilon}(r+q)\gamma_{\varepsilon x}(r-p)\gamma_{\varepsilon\varepsilon}^{\widetilde{k}_0-2}(r+q-p)\\
&+\gamma_{xx}(r)\gamma_{\varepsilon x}(r-p)\gamma_{x\varepsilon}(r+q)\gamma_{\varepsilon\varepsilon}^{\widetilde{k}_0-1}(r+q-p)\Bigg|\\
=\ &O(n^{-1}M^{2d_x+2d_\varepsilon}M^{(\widetilde{k}_0-1)(2d_\varepsilon-1)+1})+O(n^{-1}M^{2d_x+2d_\varepsilon}M^{(\widetilde{k}_0-2)(2d_\varepsilon-1)+1})\\
&+O(n^{-1}M^{2d_x}M^{d_x+d_\varepsilon}M^{(\widetilde{k}_0-1)(2d_\varepsilon-1)+1})+O(n^{-1}M^{2d_x+2d_\varepsilon}n^{2d_x-1+(\widetilde{k}_0-1)(2d_\varepsilon-1)+1})\\
&+O(n^{-1}M^{2d_x+2d_\varepsilon}n^{2d_x+2d_\varepsilon-1+(\widetilde{k}_0-2)(2d_\varepsilon-1)})+O(M^2n^{-1}n^{4d_x+2d_\varepsilon-2+(\widetilde{k}_0-1)(2d_\varepsilon-1)})\\
=\ &O\left(\frac{M^2}{n}M^{2d_x-1}M^{\widetilde{k}_0(2d_\varepsilon-1)+1}\right)+o\left(\frac{M^2}{n}M^{2d_x+2d_\varepsilon-1+\widetilde{k}_0(2d_\varepsilon-1)+1}\right)\\
&+O\left(\frac{M^2}{n}M^{3d_x-1-d_\varepsilon}M^{\widetilde{k}_0(2d_\varepsilon-1)+1}\right)+O\left(\frac{M^2}{n}M^{4d_x-1}M^{\widetilde{k}_0(2d_\varepsilon-1)+1}\right)\\
=\ &o(M^{2d_2+2d_e})\ \text{, because } 2d_e=\widetilde{k}_0(2d_\varepsilon-1)+1\ .
\end{aligned}
$$

Part III: $a\ge 3$, $\widetilde{k}_0\ge 3$

We note that, by the diagram formula (as in (16))

$$
\begin{aligned}
&\left|\operatorname{cum}\left[H_a(x_s)H_{\widetilde{k}_0}(\varepsilon_{s+p})H_a(x_{s+r})H_{\widetilde{k}_0}(\varepsilon_{s+r+q})\right]\right|\\
\le\ & C\left|\operatorname{cum}\left[H_3(x_s)H_3(\varepsilon_{s+p})H_3(x_{s+r})H_3(\varepsilon_{s+r+q})\right]\right|\ .
\end{aligned}
$$

It suffices then to focus on $a=\widetilde{k}_0=3$. There are seven different kinds of connected diagrams, which are represented in Figures 1 to 7. We have

$$
\begin{aligned}
&\sum_{p=-M}^{M}\sum_{q=-M}^{M}\frac{1}{n^2}\left|\sum_{r=-n+1}^{n-1}\sum_{s=1-r}^{n-r}\operatorname{cum}\{H_3(x_s)H_3(\varepsilon_{s+p})H_3(x_{s+r})H_3(\varepsilon_{s+r+q})\}\right|\\
=\ &\sum_{p=-M}^{M}\sum_{q=-M}^{M}\frac{C}{n^2}\Bigg|\sum_{r=-n+1}^{n-1}\sum_{s=1-r}^{n-r}\gamma_{x\varepsilon}^2(p)\gamma_{x\varepsilon}^2(q)\gamma_{\varepsilon x}(r-p)\gamma_{x\varepsilon}(r+q)+
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \gamma_{x\varepsilon}(p)\gamma_{xx}(r)\gamma_{x\varepsilon}(r+q)\gamma_{\varepsilon\varepsilon}(r+q-p)\gamma_{\varepsilon x}(r-p)\gamma_{x\varepsilon}(q)\\
&+ \quad \gamma_{xx}^2(r)\gamma_{\varepsilon\varepsilon}^2(r+q-p)\gamma_{\varepsilon x}(r-p)\gamma_{x\varepsilon}(r+q)\\
&+ \quad \gamma_{x\varepsilon}^2(p)\gamma_{x\varepsilon}^2(q)\gamma_{xx}(r)\gamma_{\varepsilon\varepsilon}(r+q-p)\\
&+ \quad \gamma_{xx}^2(r)\gamma_{\varepsilon\varepsilon}^2(r+q-p)\gamma_{x\varepsilon}(p)\gamma_{x\varepsilon}(q)\\
&+ \quad \gamma_{x\varepsilon}^2(r-p)\gamma_{x\varepsilon}^2(r+q)\gamma_{x\varepsilon}(p)\gamma_{x\varepsilon}(q)\\
&+ \quad \gamma_{x\varepsilon}^2(r-p)\gamma_{x\varepsilon}^2(r+q)\gamma_{xx}(r)\gamma_{\varepsilon\varepsilon}(r+q-p)\big|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad \frac{C}{n}\sum_{p=-M}^{M}(|p|+1)^{2(d_x+d_\varepsilon-1)}\sum_{q=-M}^{M}(|q|+1)^{2(d_x+d_\varepsilon-1)}\\
&\times \quad \Bigg[\sum_{|r|\leq 2M}(|r-p|+1)^{d_x+d_\varepsilon-1}(|r+q|+1)^{d_x+d_\varepsilon-1}\\
&+ \quad \sum_{2M<|r|\leq n}(|r-p|+1)^{d_x+d_\varepsilon-1}(|r+q|+1)^{d_x+d_\varepsilon-1}\Bigg] \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{C}{n}\sum_{p=-M}^{M}(|p|+1)^{d_x+d_\varepsilon-1}\sum_{q=-M}^{M}(|q|+1)^{d_x+d_\varepsilon-1}\\
&\times \Bigg[\sum_{|r|\leq 3M}(|r|+1)^{2d_x-1}(|r+q-p|+1)^{2d_\varepsilon-1}(|r-p|+1)^{d_x+d_\varepsilon-1}(|r+q|+1)^{d_x+d_\varepsilon-1}\\
&+ \sum_{3M<|r|\leq n}(|r|+1)^{2d_x-1}(|r+q-p|+1)^{2d_\varepsilon-1}(|r-p|+1)^{d_x+d_\varepsilon-1}(|r+q|+1)^{d_x+d_\varepsilon-1}\Bigg] \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{C}{n}\Bigg[\Bigg(\sum_{|r|\leq 3M}(|r|+1)^{2(2d_x-1)}\sum_{p=-M}^{M}\sum_{q=-M}^{M}(|r+q-p|+1)^{2(2d_\varepsilon-1)}\\
&\times(|r-p|+1)^{d_x+d_\varepsilon-1}(|r+q|+1)^{d_x+d_\varepsilon-1}\Bigg)+\sum_{p=-M}^{M}\sum_{q=-M}^{M}\Bigg(\sum_{3M<|r|\leq n}(|r|+1)^{2(2d_x-1)}\\
&\times\ |r+q-p|+1)^{2(2d_\varepsilon-1)}(|r-p|+1)^{d_x+d_\varepsilon-1}(|r+q|+1)^{d_x+d_\varepsilon-1}\Bigg)\Bigg] \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{C}{n}\sum_{p=-M}^{M}(|p|+1)^{2(d_x+d_\varepsilon-1)}\sum_{q=-M}^{M}(|q|+1)^{2(d_x+d_\varepsilon-1)}\Bigg[\Bigg(\sum_{|r|\leq 3M}(|r|+1)^{2d_x-1}\\
&\times\ (|r+q-p|+1)^{2d_x-1}\Bigg)+\sum_{3M<|r|\leq n}(|r|+1)^{2d_x-1}(|r+q-p|+1)^{2d_x-1}\Bigg] \qquad (20)
\end{aligned}
$$

$$+ \quad \frac{C}{n}\sum_{p=-M}^{M}(|p|+1)^{d_x+d_\varepsilon-1}\sum_{q=-M}^{M}(|q|+1)^{d_x+d_\varepsilon-1}\Bigg[\Bigg(\sum_{|r|\le 3M}(|r|+1)^{2(2d_x-1)}$$

$$\times(|r+q-p|+1)^{2(2d_\varepsilon-1)}\Bigg)+\sum_{3M<|r|\le n}(|r|+1)^{2(2d_x-1)}(|r+q-p|+1)^{2(2d_\varepsilon-1)}\Bigg] \quad (21)$$

$$+ \quad \frac{C}{n}\sum_{p=-M}^{M}(|p|+1)^{d_x+d_\varepsilon-1}\sum_{q=-M}^{M}(|q|+1)^{d_x+d_\varepsilon-1}\Bigg[\Bigg(\sum_{|r|\le 3M}(|r+q|+1)^{2(d_x+d_\varepsilon-1)}$$

$$\times(|r-p|+1)^{2(d_x+d_\varepsilon-1)}\Bigg)+\sum_{3M<|r|\le n}(|r+q|+1)^{2(d_x+d_\varepsilon-1)}(|r-p|+1)^{2(d_x+d_\varepsilon-1)}\Bigg] \quad (22)$$

$$+ \quad \frac{C}{n}\sum_{|r|\le 3M}(|r|+1)^{2d_x-1}\sum_{p=-M}^{M}\sum_{q=-M}^{M}(|r+p-q|+1)^{2d_\varepsilon-1}$$

$$\times(|r-p|+1)^{2(d_x+d_\varepsilon-1)}(|r+q|+1)^{2(d_x+d_\varepsilon-1)}+\sum_{3M<|r|\le n}(|r|+1)^{2d_x-1}$$

$$\times\sum_{p=-M}^{M}\sum_{q=-M}^{M}(|r+p-q|+1)^{2d_\varepsilon-1}(|r-p|+1)^{2(d_x+d_\varepsilon-1)}(|r+q|+1)^{2(d_x+d_\varepsilon-1)} \quad (23)$$

After lengthy but straightforward computations, it is not difficult to see that

$$\begin{aligned}
(17) &= O(n^{-1}M^{2d_x+2d_\varepsilon-1}M^{2d_x+2d_\varepsilon-1}M^{d_x+d_\varepsilon})+O(n^{-1}M^{4d_x-1}M^{4d_\varepsilon-1}n^{2d_x+2d_\varepsilon-1})\\
&= o\left(\frac{M^{4d_x+4d_\varepsilon-1}}{n}\right)+O(M^{4d_x-1}M^{4d_\varepsilon-1}n^{2d_x+2d_\varepsilon-2})\\
(18) &= O(n^{-1}M^{d_x+d_\varepsilon}M^{d_x+d_\varepsilon}M^{2d_x+2d_\varepsilon-1})+O(n^{-1}M^{d_x+d_\varepsilon}M^{d_x+d_\varepsilon}n^{4d_x+4d_\varepsilon-3})\\
&= O\left(\frac{M^{4d_x+4d_\varepsilon-1}}{n}\right)+o(M^{4d_x-1}M^{4d_\varepsilon-1}n^{2d_x+2d_\varepsilon-2})\\
(19) &= O(n^{-1}M^{4d_x-1}M^{4d_\varepsilon})+O(n^{-1}M^{2}n^{6d_x+6d_\varepsilon-5})\\
&= O\left(\frac{M^{4d_x+4d_\varepsilon-1}}{n}\right)+o(M^{4d_x-1}M^{4d_\varepsilon-1}n^{2d_x+2d_\varepsilon-2})\\
(20) &= O(n^{-1}M^{2d_x+2d_\varepsilon-1}M^{2d_x+2d_\varepsilon-1}M^{2d_x})+O(n^{-1}M^{4d_x-1}M^{4d_\varepsilon-1}n^{2d_x+2d_\varepsilon-1})\\
&= o\left(\frac{M^{4d_x+4d_\varepsilon-1}}{n}\right)+O(M^{4d_x-1}M^{4d_\varepsilon-1}n^{2d_x+2d_\varepsilon-2})\\
(21) &= O(n^{-1}M^{d_x+d_\varepsilon}M^{d_x+d_\varepsilon}M^{4d_\varepsilon-1})+O(n^{-1}M^{2d_x}M^{2d_x}n^{4d_x+4d_\varepsilon-3})\\
&= O\left(\frac{M^{4d_x+4d_\varepsilon-1}}{n}\right)+o(M^{4d_x-1}M^{4d_\varepsilon-1}n^{2d_x+2d_\varepsilon-2})
\end{aligned}$$

$$
\begin{aligned}
(22) &= O(n^{-1}M^{d_x+d_\varepsilon}M^{d_x+d_\varepsilon}M^{2d_x+2d_\varepsilon-1}) + O(n^{-1}M^{2d_x+2d_\varepsilon}n^{4d_x+4d_\varepsilon-3}) \\
&= O\left(\frac{M^{4d_x+4d_\varepsilon-1}}{n}\right) + o(M^{4d_x-1}M^{4d_\varepsilon-1}n^{2d_x+2d_\varepsilon-2}) \\
(23) &= O(n^{-1}M^{2d_x}M^{2d_x+2d_\varepsilon}) + O(n^{-1}M^{2}n^{6d_x+6d_\varepsilon-5}) \\
&= o(\frac{M^{4d_x+4d_\varepsilon-1}}{n}) + o(M^{4d_x-1}M^{4d_\varepsilon-1}n^{2d_x+2d_\varepsilon-2})
\end{aligned}
$$

$$
= O(\frac{M^{4d_x+4d_\varepsilon-1}}{n}) + o(M^{4d_x-1}M^{4d_\varepsilon-1}n^{2d_x+2d_\varepsilon-2}) \ .
$$

In view of the previous results, our proof will be completed if we show that

$$
\frac{M^{4d_x+4d_\varepsilon-1}}{n} + M^{4d_x-1}M^{4d_\varepsilon-1}n^{2d_x+2d_\varepsilon-2} = o(M^{2d_a+2d_e}) \ ,
$$

where

$$
2d_a + 2d_e = 2ad_x + 2\widetilde{k}_0 d_\varepsilon - (a+\widetilde{k}_0) + 2 \ .
$$

We note first that

$$
\frac{M^{4d_x+4d_\varepsilon-1}}{nM^{2d_a+2d_e}} = \frac{M^{4d_x+4d_\varepsilon-1}}{n(M^{2ad_x+2\widetilde{k}_0 d_\varepsilon-(a+\widetilde{k}_0)+2})} = \frac{M^{2(2-a)d_x+2(2-\widetilde{k}_0)d_\varepsilon-3+a+\widetilde{k}_0}}{n} \ .
$$

From (11) and $d_e > 0$ it follows that

$$
d_x > \frac{1}{2} - \frac{1}{2a} \quad \text{and} \quad d_\varepsilon > \frac{1}{2} - \frac{1}{2\widetilde{k}_0} \ ,
$$

whence, because $a, \widetilde{k}_0 \geq 2$ we have

$$
\begin{aligned}
\frac{M^{2(2-a)d_x+2(2-\widetilde{k}_0)d_\varepsilon-3+a+\widetilde{k}_0}}{n} &\leq \frac{M^{2(2-a)(\frac{1}{2}-\frac{1}{2a})+2(2-\widetilde{k}_0)(\frac{1}{2}-\frac{1}{2\widetilde{k}_0})-3+a+\widetilde{k}_0}}{n} \\
&= o\left(\frac{M^3}{n}\right) = o(1) \ ,
\end{aligned}
$$

in view of Assumption C. To complete the proof, note that, again from Assumption C, for some $\alpha > a-2, \widetilde{k}_0 - 2$ we have $M^\alpha = O(n)$, whence

$$
\frac{M^{4d_x+4d_\varepsilon-2}n^{2d_x+2d_\varepsilon-2}}{M^{2ad_x+2\widetilde{k}_0 d_\varepsilon-(a+\widetilde{k}_0)+2}} = o(M^{(2-a+\alpha)(2d_x-1)+(2-\widetilde{k}_0+\alpha)(2d_\varepsilon-1)}) = o(1) \qquad \text{as} \quad n \to \infty \ .
$$

Thus the proof is completed. □

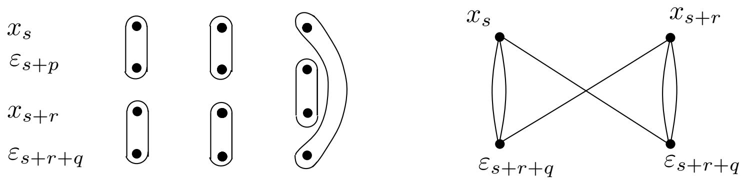

Figure 1: $\gamma^2_{x\varepsilon}(p)\gamma^2_{x\varepsilon}(q)\gamma_{x\varepsilon}(r+q)\gamma_{\varepsilon x}(r-p)$

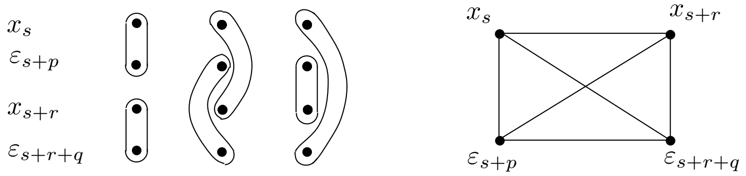

Figure 2: $\gamma_{x\varepsilon}(r+q)\gamma_{\varepsilon x}(r-p)\gamma^2_{xx}(r)\gamma^2_{\varepsilon\varepsilon}(r+q-p)$

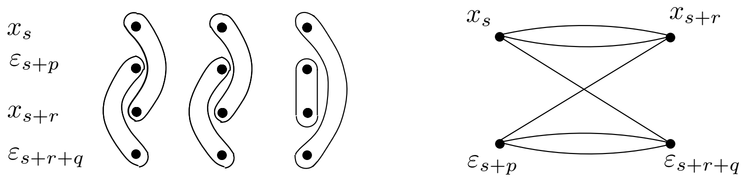

Figure 3: $\gamma_{x\varepsilon}(p)\gamma_{x\varepsilon}(q)\gamma^2_{x\varepsilon}(r+q)\gamma^2_{\varepsilon x}(r-p)\gamma_{xx}(r)\gamma_{\varepsilon\varepsilon}(r+q-p)$

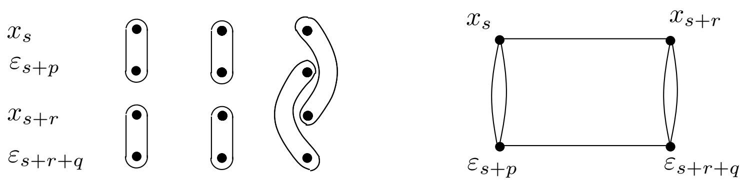

Figure 4: $\gamma^2_{x\varepsilon}(p)\gamma^2_{x\varepsilon}(q)\gamma_{xx}(r)\gamma_{\varepsilon\varepsilon}(r+q-p)$

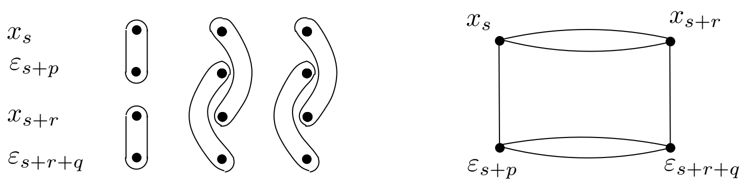

Figure 5: $\gamma^2_{x\varepsilon}(p)\gamma^2_{x\varepsilon}(q)\gamma^2_{xx}(r)\gamma^2_{\varepsilon\varepsilon}(r+q-p)$

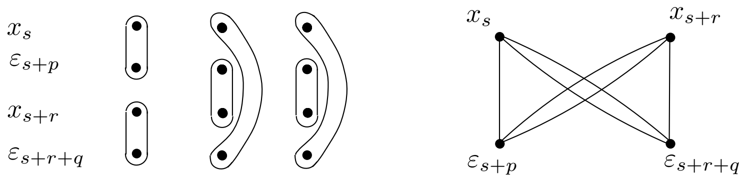

Figure 6: $\gamma_{x\varepsilon}(p)\gamma_{x\varepsilon}(q)\gamma_{x\varepsilon}^2(r+q)\gamma_{\varepsilon x}^2(r-p)$

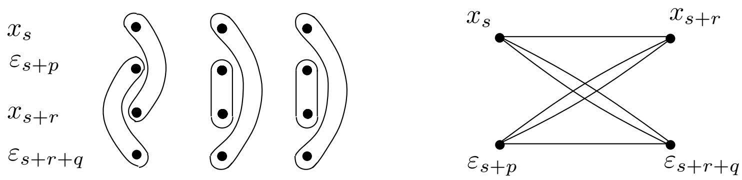

Figure 7: $\gamma_{xx}(r)\gamma_{\varepsilon\varepsilon}(r+q-p)\gamma_{x\varepsilon}^2(r+q)\gamma_{\varepsilon x}^2(r-p)$

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